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# THE EFFECT OF BUOYANCY ON THE FORMATION OF A SOLID DEPOSIT FREEZING ONTO A VERTICAL SURFACE‡

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**Abstract**—The growth rate and spatial distribution of a solid deposit freezing onto a vertical surface is analytically predicted for the case in which there is heat transfer at the moving phase interface due to natural convection in the fluid. It is shown that under certain conditions the problem may be treated as involving a transient, one-dimensional conduction process and a quasi-steady two-dimensional convection process which are coupled through the requirements of conservation of mass and energy at the phase interface. An approximate solution is obtained using integral techniques and results are in agreement with those limiting cases for which exact solutions are known.

## NOMENCLATURE

- $c$, specific heat;
- $e$, internal energy per unit mass;
- $g$, gravitational acceleration;
- $Gr$, Grashof number, $g\beta(T_\infty - T_f)L^3/\nu^2$;
- $h$, convective film coefficient;
- h, enthalpy per unit mass;
- $H$, dimensionless convective film coefficient, $h(x, t)/h(L, \infty)$;
- $k$, thermal conductivity;
- $L$, plate length;
- $\mathscr{L}$, latent heat of fusion per unit mass;
- $m$, interface velocity distribution parameter;
- $Nu$, Nusselt number, $h(L, \infty)L/k$;
- $p$, static pressure;
- $P$, dimensionless pressure, $p/p_\infty$;
- $Pr$, Prandtl number, $\nu/\alpha$;
- $q$, convective heat flux at the phase interface;
- $s$, solid phase thickness;
- $S$, interface velocity amplitude parameter defined by equation (29);
- $t$, time;
- $T$, temperature;
- $u, v$, fluid velocity components in the $x$- and $y$-directions respectively;
- $U, V$, dimensionless $u, v$ as defined by equations (14);
- $w$, dimensionless parameter defined by equation (26);
- $x, y$, Cartesian coordinates along the plate surface and normal to it respectively;
- $Z$, dimensionless distance along the plate, $(x/L)^{m+\frac{1}{4}}$;

Greek symbols

- $\alpha$, thermal diffusivity;
- $\beta$, dimensionless conduction–convection process coupling parameter, $\pi/\rho_s c(T_\infty - T_f)$;
- $\delta, \Delta$, dimensionless boundary layer thicknesses for the velocity and thermal fields respectively defined after equation (30b);
- $\eta$, dimensionless $y$-coordinate defined by equation (31);
- $\theta$, dimensionless temperature, $(T_\infty - T)/(T_\infty - T_f)$;

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$\mu$, dimensionless $y$-coordinate defined by equation (14);
$\nu$, kinematic viscosity;
$\xi$, dimensionless $x$-coordinate, $x/L$;
$\pi$, dimensionless conduction–convection process coupling parameter defined by equation (24);
$\rho$, density;
$\sigma$, dimensionless solid phase thickness, $s(x, t)/s(L, \infty)$;
$\tau$, dimensionless time, $\alpha_s t/s(L, \infty)^2$;
$\psi$, dimensionless $y$-coordinate defined by equation (31);
$\Omega$, ratio of the thermal to velocity boundary layer thickness, $\Delta/\delta$.

Vector quantities
$\nabla$, Del operator;
$\mathbf{f}$, body force per unit mass;
$\mathbf{i}, \mathbf{j}, \boldsymbol{\sigma}, \mathbf{n}$, unit vectors defined by Fig. 1;
$\boldsymbol{\tau}$, shear stress vector;
$\mathbf{W}$, fluid velocity vector.

Subscripts
$( )$, property values without a subscript refer to the fluid phase;
$( )_f$, ( ) evaluated at the phase interface;
$( )_n$, coefficient of $Z^n$ in series representation of ( );
$( )_0$, ( ) relative to the moving interface evaluated at the interface;
$( )_p$, ( ) evaluated at the plate surface;
$( )_R$, ( ) relative to the moving interface;
$( )_s$, property values of the solid phase;
$( )_x$, local, instantaneous value of ( ) based on $x$ as the reference length in dimensionless quantities;
$( )_\infty$, ( ) evaluated at a large distance from the plate.

## INTRODUCTION

THE EFFECTS of fluid motion in melting and freezing processes are of interest in a variety of applications ranging from metal processing to cryogenics and aerospace technology. Problems associated with such processes which involve simultaneous heat transfer and phase change and where both the temperature distribution and the motion of the phase interface must be determined are commonly referred to as "Stefan-like" problems. In general, the system of equations describing such phenomena is non-linear. With the exception of a few cases [1] involving simple boundary and initial conditions, no exact solutions have been obtained. For more complex situations, a variety of numerical, analog, and approximate solutions have appeared. A detailed review of the literature has been published by Muehlbauer and Sunderland [2].

Analyses considering the effects of convection on the freezing process have appeared only within the last few years. A brief review of these efforts is presented in [3]. Included are several approximate solutions [4–7] for determining the effect of convection near the moving phase interface. These all assume that the convective heat flux is known in advance. However, no general method for determining this flux was available. One objective of the present investigation, therefore, was to obtain an analytic estimate for the rate of convective heating due to natural convection at the phase interface in the case of a solid being deposited on a vertical, isothermal surface.

In comparison with times of interest in melting and freezing processes, natural convective transients are often of negligible duration. From the point of view of an observer moving with the phase interface however the fluid would appear to be moving into the solid surface at a time varying rate. It is well known [8] that large changes in the rate of convective heat transfer to a surface can be produced by small rates of suction at that surface. Motivated by these considerations, a dimensional analysis of the complete set of governing equations was undertaken. The results show that the freezing or melting process may be analyzed as involving transient conduction in the solid phase and quasi-steady convection in the fluid as viewed by an observer moving with the phase interface.

The solution obtained for the conduction process involves a parameter which must be specified by analysis of the convection process or by experiment. Using the conduction solution however the shape of the suction velocity distribution along the phase interface could be determined. Based on this result, a von Karman integral analysis of the convection process is performed. Combining the results of the convection and conduction solutions, quantitative predictions of heat transfer and solidification rates are obtained.

## MATHEMATICAL FORMULATION

Consider a flat, vertical plate of length $L$ immersed in an infinite fluid initially at a uniform temperature, $T_\infty$. For time greater than zero let the temperature of the plate have the uniform and constant value $T_p$ which is assumed to be less than the solidification temperature $T_f$ of the surrounding fluid. Due to the inertia of the fluid and the finite propagation velocity of disturbances associated with convective processes [9, 10], a one dimensional conduction solution may be expected to apply over most of the plate for sufficiently small times. Eventually however, as a result of the spatial dependence of the convective heat flux at the phase interface, a two dimensional solid deposit forms on the surface of the plate as shown schematically in Fig. 1.

Neglecting viscous dissipation and compression work, assuming that all properties (with the exception of the fluid density) are uniform and constant in each phase, that the solid adheres to the plate, and that a smooth phase phase boundary is formed, the following conservation equations describe transient, two dimensional melting or freezing with natural convection in the fluid phase.

$$\frac{\partial T_s}{\partial t} = \alpha_s \nabla^2 T_s \quad \begin{matrix} 0 < x < L \\ 0 \leqslant y < s \end{matrix} \tag{1}$$

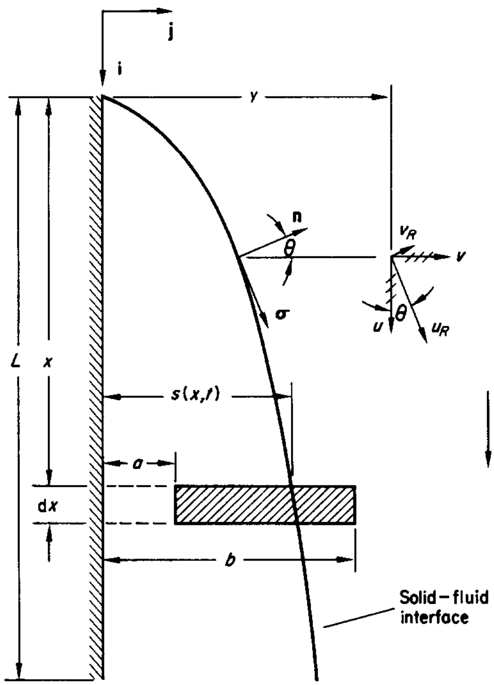


FIG. 1. Schematic representation of the solidification process and coordinate definitions.

$$\left.\begin{aligned} &\nabla . \mathbf{W} = 0 &(2)\\ &\frac{D\mathbf{W}}{Dt} = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{W} &(3)\\ &\frac{DT}{Dt} = \alpha \nabla^2 T &(4)\end{aligned}\right\} \quad y > s.$$

Here $\mathbf{W}$ is the fluid velocity vector, $\mathbf{f}$ is the body force vector per unit mass and the subscript $s$ denotes properties of the solid phase. It has been assumed that the fluid is Newtonian and there are no sources or sinks in either phase. These equations together with an equation of state for the fluid and appropriate boundary and initial conditions completely describe the phenomenon.

The unique feature of Stefan-like problems is associated with the requirements of conservation of mass and energy at the moving phase interface. These boundary conditions may be derived from the control volume forms of the continuity and energy equations.

$$\frac{d}{dt}\iiint \rho \, d(\text{vol}) + \iint \rho \mathbf{W} . \mathbf{n} \, dA = 0 \tag{5}$$

$$\frac{d}{dt}\iiint \rho\left(e + \frac{W^2}{2}\right) d(\text{vol}) + \iint \rho\left(e + \frac{W^2}{2} + \frac{p}{\rho}\right)\mathbf{W} . \mathbf{n} \, dA = \iiint \rho \mathbf{f} . \mathbf{W} \, d(\text{vol}) + \iint \boldsymbol{\tau} . \mathbf{W} \, dA + \iint k\nabla T . \mathbf{n} \, dA. \tag{6}$$

Here $\tau$ is the shear stress vector acting on the control surface and $\mathbf{n}$ is a unit outward normal. When applied to the control volume shown as the shaded region in Fig. 1 and taking the limit as $a$ and $b$ approach $s$, the following boundary conditions are obtained at $y = s$.

$$(\rho - \rho_s)\frac{\partial s}{\partial t} = \rho\left(v - u\frac{\partial s}{\partial x}\right) \tag{7}$$

$$\rho_s(h_{0s} - h_0)\frac{\partial s}{\partial t} = k\frac{\partial T}{\partial y} - k_s\frac{\partial T_s}{\partial y} + \frac{\partial s}{\partial x} \times \left\{k\frac{\partial T}{\partial x} - k_s\frac{\partial T_s}{\partial x}\right\} + \rho v\left(u - v\frac{\partial s}{\partial x}\right) \times \left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}. \tag{8}$$

The stagnation enthalpy difference $(h_0 - h_{0s})$ is the latent heat of fusion, $\mathscr{L}$.

In order to specify both fluid velocity components at the phase interface, an additional boundary condition is required. This may be obtained by noting that the velocity of the fluid tangent to the phase interface must be zero, i.e.

$$u + v\frac{\partial s}{\partial x} = 0 \quad \text{at} \quad y = s. \tag{9}$$

Note that if $\rho = \rho_s$ and $\partial s/\partial x = 0$, equations (7), (8) and (9) reduce to the form

$$\left.\begin{aligned} &u = v = 0 & (10)\\ &\rho_s\mathscr{L}\frac{ds}{dt} = k_s\frac{\partial T_s}{\partial y} - k\frac{\partial T}{\partial y} & (11)\end{aligned}\right\} \text{ at } y = s.$$

Equation (11) is the classical "Stefan boundary condition."

## DIMENSIONAL ANALYSIS

Having thus arrived at a general mathematical description of two dimensional melting or freezing phenomena, the next step is to determine whether any terms in the general equations (1–4), (7–9) are unimportant for the special case under consideration. This may be done by putting the equations into a dimensionless form such that for the situation of interest the values of all dimensionless derivatives are the same order of magnitude. The success of such an analysis depends very strongly on the degree to which specific phenomena are understood. For the present problem, the greatest difficulty was in the choice of the appropriate dimensionless time parameter.

It is known [9–11] that natural convection transients are generally of short duration relative to times of interest in melting or freezing processes. Therefore, the characteristic time for the overall process was chosen by considering transport processes in the solid and at the phase interface. For constant temperatures at all boundaries, $\partial s/\partial t$ must approach zero as time becomes large. Therefore the conductive heat efflux from the phase interface must approach the convective heating rate. Equating these quantities and considering situations in which $\partial s/\partial x$ is small, the following relation is obtained

$$h(x, \infty)(T_f - T_\infty) + k_s(T_f - Y_p)/s(x, \infty) = 0. \tag{12}$$

Here $h(x, \infty)$ is the local, steady state, convective film coefficient. Therefore, in contrast to classical conduction solutions [1], the presence of an infinite fluid medium in which convection may occur implies the existence of a maximum, finite, limiting thickness of the solid layer given by

$$s(L, \infty) = \frac{k_s(T_f - T_p)}{h(L, \infty)(T_\infty - T_f)}. \tag{13}$$

This quantity is a natural characteristic length determined by material properties, initial and boundary conditions. Using this result, it is shown in [3] that when the ratio $c_s(T_f - T_p)/\mathscr{L}$ is less than unity, the appropriate dimensionless time parameter is

$$\tau = \alpha_s t/s(L, \infty)^2.$$

Thus, the following dimensionless quantities may be introduced.

$$\left.\begin{aligned}
&\xi = x/L & &\tau = \alpha_s t/s(L, \infty)^2 \\
&\mu_s = y/s(L, \infty) & &\mu = (Pr^2 Gr)^{\frac{1}{4}} y/L \\
&\sigma = s/s(L, \infty) & &P = p/p_\infty \\
&\theta_s = (T_s - T_p)/(T_f - T_p) & &\theta = (T_\infty - T)/(T_\infty - T_f) \\
&H = h(x, t)/h(L, \infty) & &U = (Pr^2 Gr)^{-\frac{1}{2}} uL/\alpha \\
&Pr = \nu/\alpha & &V = (Pr^2 Gr)^{-\frac{1}{4}} vL/\alpha \\
& & &Gr = g\beta(T_\infty - T_f)L^3/\nu^3
\end{aligned}\right\} \quad (14)$$

Now considering those cases in which

$$\left.\begin{aligned}
&\frac{s(L, \infty)}{L}^{2} \leqslant \varepsilon \ll 1 \\
&\frac{\alpha_s}{\alpha}\left\{\frac{L}{s(L, \infty)}\right\}^2 \{Pr^2 Gr\}^{-\frac{1}{2}} \leqslant \varepsilon
\end{aligned}\right\} \quad (15)$$

and omitting terms multiplied by coefficients of order $\varepsilon$, equations (1–4) and (7–9) reduce to the following.

$$\frac{\partial \theta_s}{\partial \tau} = \frac{\partial^2 \theta_s}{\partial \mu_s^2} \qquad 0 \leqslant \mu_s < \sigma \quad (16)$$

$$\left.\begin{aligned}
&\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \mu} = 0 & (17a)\\
&U\frac{\partial U}{\partial \xi} + V\frac{\partial U}{\partial \mu} = \theta + Pr\frac{\partial^2 U}{\partial \mu^2} & (17b)\\
&\partial P/\partial \mu = 0 & (17c)\\
&U\frac{\partial \theta}{\partial \xi} + V\frac{\partial \theta}{\partial \mu} = \frac{\partial^2 \theta}{\partial \mu^2} & (17d)
\end{aligned}\right\} \mu_s > \sigma.$$

$$\left.\begin{aligned}
&V = \frac{\alpha_s}{\alpha}\left\{\frac{L}{s(L, \infty)}\right\}(Pr^2 Gr)^{-\frac{1}{4}}\left\{1 - \frac{\rho_s}{\rho}\right\}\frac{\partial \sigma}{\partial \tau} & (18)\\
&\frac{\mathscr{L}}{c_s(T_f - T_p)}\frac{\partial \sigma}{\partial \tau} + H + \frac{\partial \theta_s}{\partial \mu_s} = 0 & (19)\\
&U = 0 & (20)
\end{aligned}\right\} \mu_s = \sigma.$$

Equations (17) are the classical [12] boundary layer equations for steady state natural convection. Here, however, their solution is complicated by their coupling with the conduction equations through (18) and (19), and by the fact that the boundary of the fluid region is described by an unknown function, $\sigma(\xi, \tau)$.

## VARIABLE TRANSFORMATION

The latter difficulty can be eliminated by introducing coordinates fixed relative to the phase interface and by writing the governing equations for the velocity and temperature fields relative to this moving system. That is, referring to Fig. 1, let

$$t_R = t; \quad \mathbf{W} = u\mathbf{i} + v\mathbf{j}; \quad X_R = x;$$

$$Y_R = y - s; \quad \mathbf{W}_R = \mathbf{W} - \frac{\partial s}{\partial t}\mathbf{j} = u_R\boldsymbol{\sigma} + v_R\mathbf{n}. \quad (21)$$

The subscript $R$ refers to quantities relative to the phase interface. Introducing (21) into the governing equations (17, 18) for the convection process, the following equations are obtained.

$$\left.\begin{aligned} &\frac{\partial U_R}{\partial \xi_R} + \frac{\partial V_R}{\partial \mu_R} = 0 && (22a)\\ &U_R\frac{\partial U_R}{\partial \xi_R} + V_R\frac{\partial U_R}{\partial \mu_R} = \theta + Pr\frac{\partial^2 U_R}{\partial^2 \mu_R^2} && (22b)\\ &\frac{\partial P}{\partial \mu_R} = 0 && (22c)\\ &U_R\frac{\partial \theta}{\partial \xi_R} + V_R\frac{\partial \theta}{\partial \mu_R} = \frac{\partial^2 \theta}{\partial \mu_R^2} && (22d) \end{aligned}\right\} \mu_R > 0$$

$$V_R = -\frac{\alpha_s}{\alpha}\frac{L}{s(L,\infty)}(Pr^2 Gr)^{-\frac{1}{4}}\frac{\rho_s}{\rho}\frac{\partial \sigma}{\partial \tau_R}; \quad \mu_R = 0 \quad (23)$$

Here the subscript $R$ implies that $u_R$, $v_R$, $X_R$ and $Y_R$ have been substituted for $u$, $v$, $x$ and $y$ in equation (14). In this new coordinate system, the boundary of the fluid region corresponds to $\mu_R = 0$. The term quasi-steady is used to describe the associated convective phenomena since time appears as a parameter in equation (23).

## APPROXIMATE SOLUTION

Equations (16), (19, 20) and (22, 23) represent a mathematical model which includes natural convection in the fluid phase and describes the formation of a solid on a vertical surface. An approximate solution for (16) and (19) has appeared [7] based on the assumption that the local convective heat flux at the phase interface and the interface velocity are related by

$$q(x,t) = q(x,\infty) + \pi\frac{\partial s}{\partial t}. \quad (24)$$

Here $\pi$ is a coefficient to be evaluated from knowledge of the convective process in the fluid phase. The solution obtained in [7] may be written as

$$(1 - \sigma\xi^{-\frac{1}{4}})\exp(\sigma\xi^{-\frac{1}{4}}) = \exp(-\{\tau\xi^{-\frac{1}{2}}/f(w)\}) \quad (25)$$

where

$$\left.\begin{aligned} f(w) &= \frac{15 + 10w + 2w^2}{5w(3 + w)}\\ w &= \frac{c_s(T_f - T_p)}{\mathscr{L}(1 + \pi/\rho_s\mathscr{L})}. \end{aligned}\right\} \quad (26)$$

Equation (25) is plotted in Fig. 2 where Neumann's pure conduction solution [1] is also shown. Comparison of these results implies that convective effects are unimportant for

$$\sigma\xi^{-\frac{1}{4}} < 0{\cdot}3. \quad (27)$$

Using dimensionless variables, the solid thickness ($\sigma$) corresponding to equation (25) is shown as a function of position ($\xi$) along the plate with time $[\tau/f(w)]$ as a parameter in Fig. 3. Based on this solution, the phase interface velocity may be determined in terms of the parameter $\pi$ (or the dimensionless parameter $w$). That is, using equation (25) in (23)

$$V_0 \equiv V_R|_{\mu_R=0} = -\frac{c(T_\infty - T_f)}{c_s(T_f - T_p)}\frac{Nu}{(Pr^2 Gr)^{\frac{1}{4}}} \times \left\{\frac{1}{\sigma} - \xi^{-\frac{1}{4}}\right\}\Big/ f(w), \quad (28)$$

where $Nu = h(L,\infty)L/k$. $V_0$ is shown as a function of $\xi$ with $\tau/f(w)$ as a parameter in Fig. 4. This figure shows that at any instant of time, the interface velocity distribution may be approximated by

$$V_0 \approx -S\xi^m \quad (29)$$

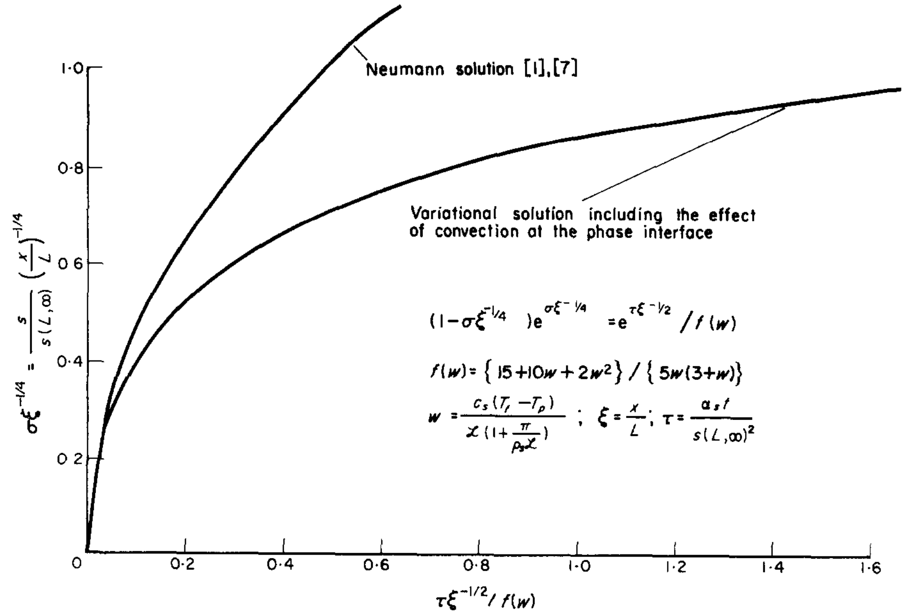


FIG. 2. Comparison of the variational solution including the effect of convection with Neumann's pure conduction solution.

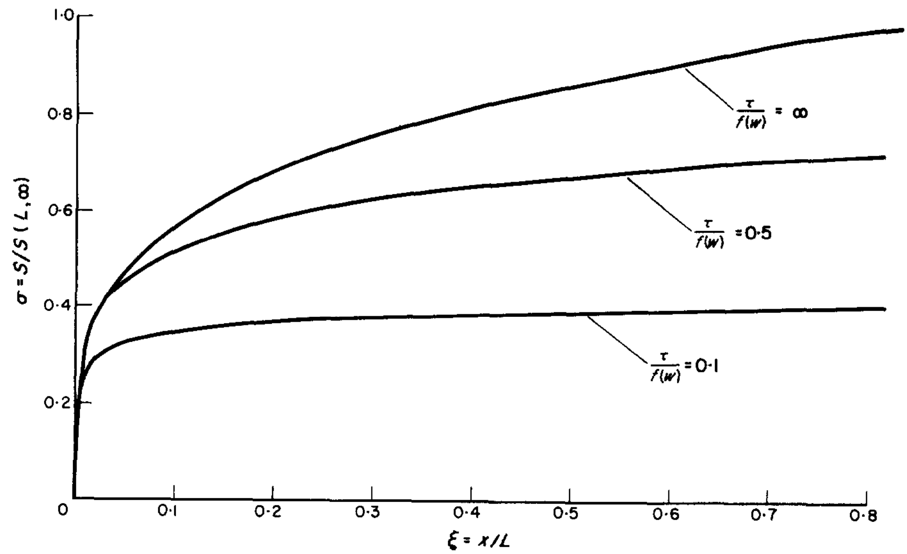


FIG. 3. Dimensionless solid thickness vs. distance from the leading edge of the plate with time as a parameter.

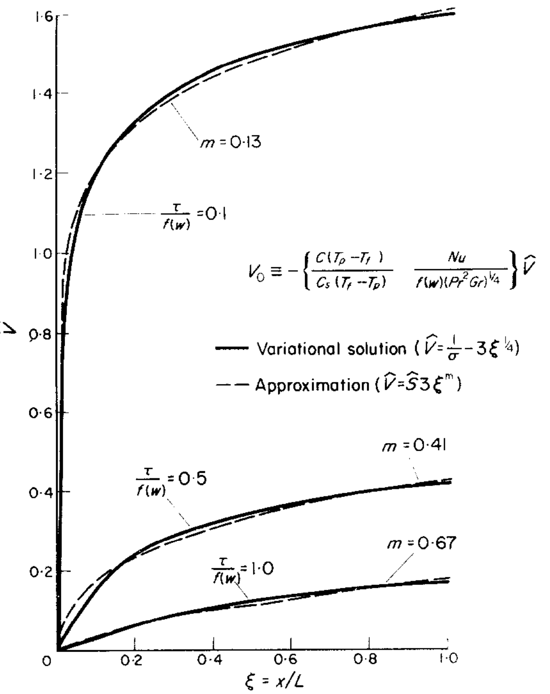


FIG. 4. Comparison of the variational solution for the interface velocity distribution with a simpler approximating function.

where $S$ and $0 < m < 1$ are functions of $\tau/f(w)$. This approximation (29) can now be used to simplify the boundary condition (23) required for solution of the equations (22) describing the quasi-steady convective process. That solution will then be used to determine the validity of the assumed relation (24) leading to the conduction solution and will furthermore permit evaluation of the parameter $\pi$.

## INTEGRAL ANALYSIS OF THE QUASI-STEADY CONVECTION PROCESS

The quasi-steady convection process may be analysed using von Karman's integral technique. The velocity distribution along the boundary of the fluid region relative to the moving phase interface is approximately described by equation (29). Integrating the boundary layer equations (22) with respect to $\mu_R$ and using continuity to express $V_R$ in terms of $U_R$, the von Karman equations are obtained.

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\delta\int_0^1 U_R^2\,\mathrm{d}\eta\right\} + \frac{Pr}{\delta}\frac{\partial U_R}{\partial\eta}\bigg|_{\eta=0} = \Delta\int_0^1 \theta\,\mathrm{d}\psi \quad (30a)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left\{\Delta\int_0^1 \theta U_R\,\mathrm{d}\psi\right\} + \frac{1}{\Delta}\frac{\partial\theta}{\partial\psi}\bigg|_{\psi=0} = V_0. \quad (30b)$$

Here $\delta$ and $\Delta$ are the values of $\mu_R$ at the edges of the momentum and thermal boundary layers in the fluid and

$$\eta = \mu_R/\delta, \qquad \psi = \mu_R/\Delta. \quad (31)$$

Representing $U_R$ and $\theta$ by the polynomials

$$U_R = \frac{\delta^2}{6Pr + \delta V_0}\eta(1-\eta)^3$$

$$\theta = 1 - \frac{6}{6 + \Delta V_0}\psi(2 - 2\psi^2 + \psi^3) - \frac{\Delta V_0}{6 + \Delta V_0}\psi^2(6 - 8\psi + 3\psi^2), \quad (32)$$

equations (30) become ordinary differential equations for $\delta$ and $\Delta$ as functions of $\xi$. The profiles chosen (32) satisfy the energy and momentum equations identically at $\mu_R = 0$. By using equation (29) for $V_0$, (32) for $U_R$, $\theta$ and defining

$$Z \equiv \xi^{m+\frac{1}{4}} \quad (33)$$

equations (30) are transformed into a pair of simultaneous differential equations that are amenable to power series analysis. That is, introducing the following power series representations

$$\delta = \xi^{\frac{1}{4}}\sum_{n=0}^{\infty}\delta_n Z^n \quad (34)$$

$$\Omega = \sum_{n=0}^{\infty}\Omega_n Z^n, \quad (35)$$

where

$$\Omega = \Delta/\delta \quad (36)$$

into the resulting equations, and equating coefficients of like powers of $Z$, algebraic equations are obtained for the coefficients $\delta_n$, $\Omega_n$. The resulting algebraic equations can then be solved explicitly. The details of this somewhat lengthy procedure are described in [3]. Some of the manipulations involved imply that the solution must be limited to the range.

$$-1 < \frac{\delta V_0}{6Pr} < 1 \quad \text{and} \quad -\tfrac{1}{2} < \frac{\Delta V_0}{6} < 1. \qquad (37)$$

Unfortunately, the authors have been unable to rigorously demonstrate the convergence of the resulting series. Up to thirty pairs of coefficients ($\delta_n$, $\Omega_n$) have been evaluated however, and within the range described by (37) the ratio of successive coefficients remains less than one. Computations were terminated when the last computed coefficients changed the value of $\Delta$ at $\xi = 1$ by less than 0·1 per cent. Once the $\delta_n$ and $\Omega_n$ have been evaluated, the local convective heat transfer coefficient can be determined from the solution for the temperature distribution. This results in the following dimensionless relation.

$$\frac{Nu_x}{Gr_x^{\frac{1}{4}}} = \frac{2\xi^{\frac{1}{4}}\sqrt{Pr}}{\Delta(1 + \Delta V_0/6)}. \qquad (38)$$

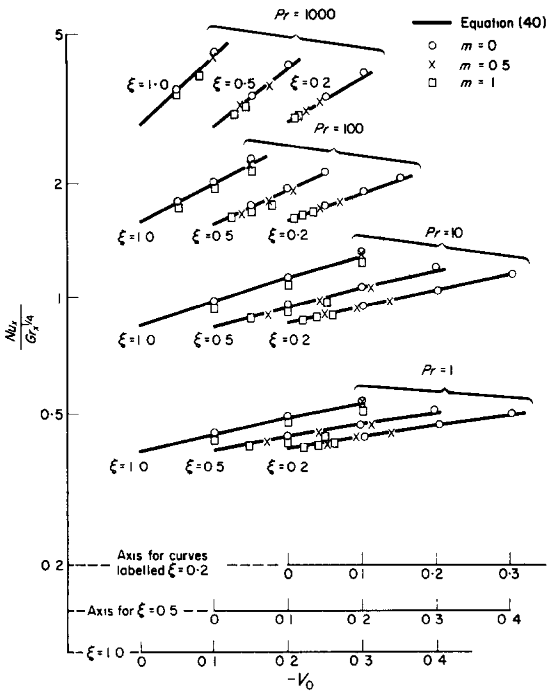


FIG. 5. Comparison of the heat flux at the phase interface with a linear approximation.

## RESULTS

Computed values of $Nu_x/Gr_x^{\frac{1}{4}}$ are shown as points in Fig. 5 as a function of the local magnitude of the relative fluid velocity ($V_0$) at the phase interface, $Pr$ and $m$. This figure shows that the local convective heat flux parameter, $Nu_x/Gr_x^{\frac{1}{4}}$, is primarily a function of the local magnitude of $V_0$ and $Pr$. Dependence on the boundary velocity distribution parameter, $m$, is of secondary importance for $0 < m < 1$.

For the case of $m = 0$, $Pr = 0{\cdot}72$ a first order perturbation analysis for natural convection with suction or injection has been obtained by Sparrow and Cess [8]. A comparison of their results with the present power series analysis is shown in Fig. 6. Figure 7 provides a comparison with Ostrach's solution [13] for the case of $V_0 = 0$.

The present solution is based on the assumption that the local convective heat flux and the

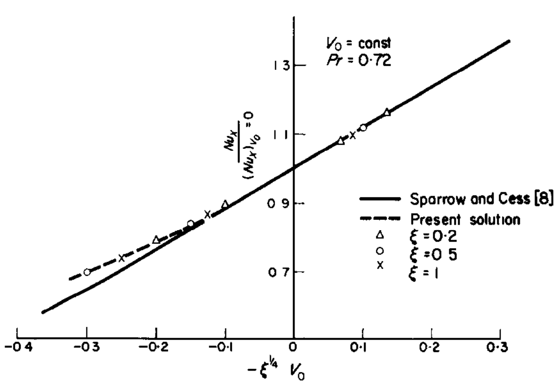


FIG. 6. Comparison with a first order perturbation analysis for constant $V_0$.

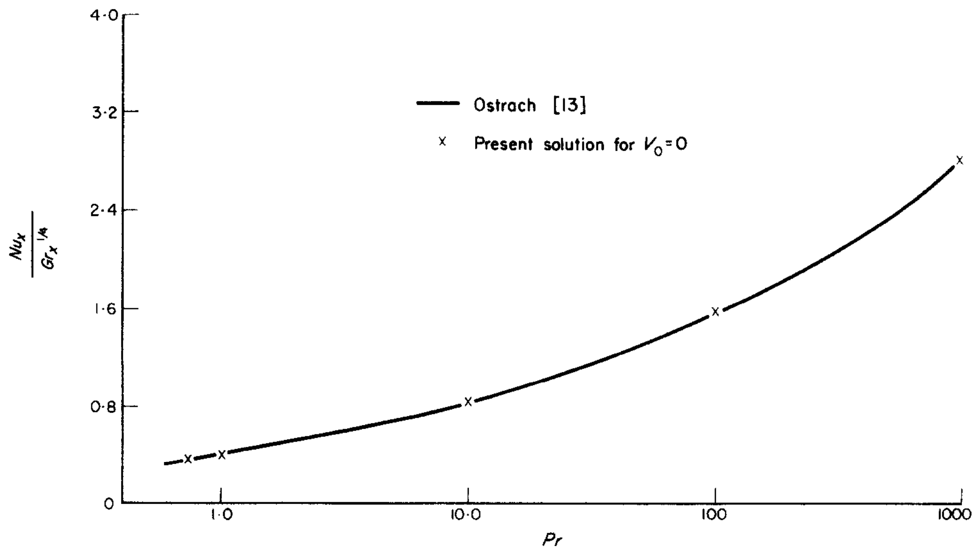


FIG. 7. Comparison with Ostrach's results for $V_0 = 0$.

local phase interface velocity are related by equation (24). Introducing a new parameter, $\beta$, such that

$$\frac{\pi}{\rho_s \mathscr{L}} = \beta \frac{c(T_\infty - T_f)}{\mathscr{L}} \tag{39}$$

equation (24) may be rewritten in dimensionless form as

$$\frac{Nu_x}{Gr_x^{\frac{1}{4}}} = \frac{Nu(x, \infty)}{Gr_x^{\frac{1}{4}}} - \beta \sqrt{Pr}\, \xi^{\frac{1}{4}} V_0. \tag{40}$$

Values of $\beta$ determined from a least square error fit of the results of the power series analysis are shown in Fig. 8. Using the resulting values of $\beta$, equation (40) is represented by the solid curves in Fig. 5. Comparison of these curves with the data points also shown indicates that the assumed linear relation (24) is quite reasonable in the range of validity of the power series solution.

Using Fig. 8, $\beta$ can be determined for a given set of conditions. Equations (25, 26), (28) and (39, 40) then represent a complete solution for solidification onto a vertical surface with natural convection in the fluid phase. The validity of this solution is limited to the range indicated by (37). Using equation (28), it can be shown that this is equivalent to the limitation

$$\frac{\xi^{\frac{1}{4}}}{\sigma} < 1 + \frac{3}{2} f(w) \frac{c_s(T_f - T_p)}{c(T_\infty - T_f)}. \tag{41}$$

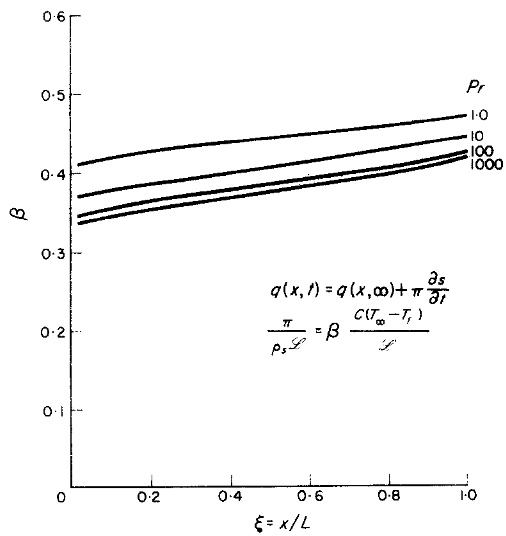


FIG. 8. The conduction–convection coupling parameter $\beta$ as a function of position and Prandtl number.

In many cases this range of validity overlaps that of the pure conduction solution (27).

### NUMERICAL EXAMPLE

As a specific example to illustrate the application of the solution obtained here consider a flat, vertical plate at a temperature of $-40°F$ (for time greater than zero) immersed in a water environment with $T_\infty = 62°F$. Then letting $L = 1$ ft, $T_\infty - T_f = 30°F$, $T_f - T_p = 72°F$ and using the property values given in [14]

$$Pr = 9{\cdot}85$$

and

$$Gr = g\beta(T_\infty - T_f)L^3/\nu^2 = 1{\cdot}92 \times 10^8$$

therefore from Fig. 7

$$Nu/Gr^{\frac{1}{4}} = 0{\cdot}83$$

so that

$$Nu = 97{\cdot}3$$

and

$$\frac{s(L, \infty)}{L} = \frac{k_s(T_f - T_p)}{NuLk(T_\infty - T_f)} = 0{\cdot}0955$$

i.e.

$$s(L, \infty) = 1{\cdot}15 \text{ in.}$$

Next the following values are obtained for $\beta$ from Fig. 8

| $\xi$ | $\beta$ |
|---|---|
| 0·1 | 0·38 |
| 0·5 | 0·41 |
| 1·0 | 0·45 |

Therefore at $\xi = 0{\cdot}5$

$$\frac{\pi}{\rho_s\mathscr{L}} = \beta\frac{c(T_\infty - T_f)}{\mathscr{L}} = 0{\cdot}085$$

$$w = \frac{c_s(T_f - T_p)/\mathscr{L}}{1 + (\pi/\rho_s\mathscr{L})} = 0{\cdot}216$$

$$f(w) = \frac{15 + 10w + 2w^2}{5w(3 + w)} = 4{\cdot}97.$$

The thickness of the solid is determined using the logarithm of equation (25)

$$\frac{\tau\xi^{-\frac{1}{2}}}{f(w)} = -\sigma\xi^{-\frac{1}{4}} - \ln\{1 - \sigma\xi^{-\frac{1}{4}}\}$$

For example with

$$\sigma\xi^{-\frac{1}{4}} = 0{\cdot}9$$

$$\tau\xi^{-\frac{1}{2}}/f(w) = 1{\cdot}4,$$

or

$$\tau = 5{\cdot}85$$

and

$$t = s(L, \infty)^2\tau/\alpha_s = 1{\cdot}12 \text{ h}.$$

The thickness of the solid phase at this instant in time and position along the plate is

$$s = 0{\cdot}9\, s(L, \infty)\xi^{\frac{1}{4}} = 0{\cdot}868 \text{ in.}$$

To examine the validity, in this particular example, of the assumptions used in the analysis leading to this solution, the coefficients given by equation (15) must be evaluated. In this case

$$\frac{s(L, \infty)}{L}^2 = 0{\cdot}0092 \ll 1$$

$$\frac{\alpha_s}{\alpha}\left(\frac{L}{s(L, \infty)}\right)^2 (Pr^2Gr)^{-\frac{1}{4}} = 0{\cdot}0073 \ll 1.$$

Finally, to determine whether the ranges of validity of the approximate quasisteady convection solution and the Neumann conduction solution overlap, equation (41) is utilized. The Neumann solution [1] cannot be distinguished from the present solution for $\sigma\xi^{-\frac{1}{4}} < 0{\cdot}3$. From equation (41) the present solution may be expected to be valid for

$$\sigma\xi^{-\frac{1}{4}} > \left\{1 + \frac{3f(w)}{2}\frac{c_s(T_f - T_p)}{c(T_\infty - T_f)}\right\}^{-1} = 0{\cdot}12.$$

Thus, the ranges of validity of the two solutions do overlap so that, in this example for all practical purposes the present solution may be used during the entire freezing process.

With regard to this example it is of course recognized that water is a peculiar substance in that its density has a maximum at 4°C. Therefore the profiles used in the integral analysis of the convective process undoubtedly introduce some error when applied to this case. It was felt, however, that this disadvantage was offset by the ready availability of all the property data required to illustrate the solution. For many materials adequate property information is quite difficult to obtain.

Another characteristic of water is that it has a rather high heat of fusion. Therefore in the example considered the quantity $\pi/\rho_s\mathscr{L}$ was found to be relatively small—i.e. of the order of 10 per cent. Thus, if the convective heating rate at the phase interface was assumed to be identically equal to the steady state value, $q(x, \infty)$, no more than a 10 per cent error* is introduced in estimates of the time required to reach a given thickness of solid.

On the other hand, if the solidification of liquid hydrogen (a process used in the purification of rocket fuel) is considered then according to [14, 15]

$$\mathscr{L} = 14 \text{ cal/g}$$

and with $T_\infty - T_f = 10°\text{K}$,

$$c = 2{\cdot}1 \text{ cal/g°K}$$

$$Pr = 1{\cdot}25.$$

Therefore, from Fig. 8, $\beta \approx 0{\cdot}44$ and

$$\frac{\pi}{\rho_s\mathscr{L}} = \beta\frac{c(T_\infty - T_f)}{\mathscr{L}} = 0{\cdot}66.$$

Thus, in this case the time required to produce a given thickness of solid would be considerably underestimated if the effect of suction at the phase interface is ignored.

* This is due to the fact that $\pi > 0$ and $f(w) \sim 1/w$ for $w < 1$.


**SUMMARY**

The growth rate and spatial distribution of a solid deposit freezing onto a vertical surface has been predicted based on considerations including the complete set of two dimensional, transient equations and natural convection in the fluid phase. Due to the extreme complexity of such phenomena a simple set of boundary conditions and a series of simplifying assumptions were introduced to reduce the mathematical description to tractable form. It has been shown that under certain conditions freezing or melting phenomena may be treated as involving a transient, one-dimensional conduction process and a quasi-steady, two-dimensional convection process coupled through the requirements of conservation of mass and energy at the moving phase interface. Assuming a linear relation between the convective heating rate at the phase interface and the interface velocity, it became possible to effectively decouple the two processes. The equations describing each process were then solved approximately using integral techniques. The solution for the convective process verified the validity of the assumed linear relation used to decouple the segments of the overall problem. The two solutions when combined then provide a solution for the complete problem.

The range of validity of the results is limited by various assumptions used in the analysis. These limitations are defined by equations (15). Furthermore, several mathematical manipulations introduce the limitations of equations (37) or (41). Laminar, natural convection boundary layer flow has been assumed and results have been obtained for Prandtl numbers greater than 0·7 only.


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## EFFET DE LA FLOTTABILITÉ SUR LA FORMATION D'UN DÉPÔT SOLIDE EN TRAIN DE SE CONGELER SUR UNE SURFACE VERTICALE

**Résumé**—La vitesse de croissance et la distribution spatiale d'un dépôt solide qui se congèle sur une surface verticale sont prédites analytiquement dans le cas où il y a transport de chaleur à l'interface mobile entre des phases dû à la convection naturelle dans le fluide. On montre que sous certaines conditions le problème peut être traité comme impliquant un processus de conduction transitoire et unidemensionnel et un processus de convection quasi-permanent et bidimensionnel qui sont couplés à travers les conditions de conservation de la masse et de l'énergie à l'interface entre les phases. Une solution approchée est obtenue en employant des techniques intégrales et les résultats sont en accord avec les cas limites pour lesquels on connaît des solutions exactes.

## DER EINFLUSS DES AUFTRIEBS AUF DIE BILDUNG EINER GEFRIERENDEN FESTABLAGERUNG AN EINE SENKRECHTE OBERFLÄCHE

**Zusammenfassung**—Die Wachstumsrate und die räumliche Verteilung einer festen Schicht, die an einer senkrechten Wand anfriert, wird analytisch bestimmt für den Fall, dass an der wandernden Phasengrenze der Wärmeübergang durch die natürliche Konvektion der Flüssigkeit bestimmt wird. Es wird gezeigt, dass unter gewissen Bedingungen dieses Problem behandelt werden kann wie ein instationärer eindimensionaler Wärmeleitungsvorgang und eine quasistationäre zweidimensionale Konvektionsströmung, welche durch die Bedingung der Masse- und Energieerhaltung an der Phasengranze gekoppelt sind.

Mit Hilfe von Integralverfahren wird eine Näherungslösung angegeben, die Ergebnisse stimmen mit den exakten Lösungen, die für Grenzfälle bekannt sind, überein.

## ВЛИЯНИЕ ПЛОВУЧЕСТИ НА ОБРАЗОВАНИЕ ТВЕРДОГО ОСАДКА, НАМЕРЗАЮЩЕГО НА ТВЕРДОЙ ВЕРТИКАЛЬНОЙ ПОВЕРХНОСТИ

**Аннотация**—Скорость роста и распределение в пространстве осадка, затвердевающего вследствие кристаллизации на вертикальной поверхности, рассчитывается аналитически для случая переноса тепла на движущейся вследствие естественной конвекции в жидкости границе раздела фаз. Показано, что при определенных условиях может быть использована математическая модель, описыьающая переходный одномерный процесс теплопроводности и квазистационарный двумерный процесс конвекции, которые объединяются в силу требований сохранения массы и энергии на границе раздела фаз. Путем интегральных преобразований получено приближенное решение. Результаты согласуются с предельными случаями, для которых известны точные решения.